4. L. Debnath and S. Rosenblat, "The ultimate approach to the steady state in the generation of waves on a running stream," Quart. J. Mech. Appl. Math., 22, No. 2 (1969).
5. L. V. Cherkesov, Surface and Internal Waves [in Russian], Naukova Dumka, Kiev (1973).
6. A. K. Pramanik, "Waves due to a moving oscillatory surface pressure in a stratified fluid," Trans. ASME, Ser. E. J. Appl. Mech., 41, No. 3 (1974).
7. A. K. Pramanik, "Generation of internal gravity waves in a stratified ocean," Bull. Cal. Math. Soc., 75, No. 1 (1983).
8. F. Tricomi, Differential Equations [Russian translation], IL, Moscow (1962).
9. V. V. Goncharov, "Some features of internal waves in the ocean," in: Tsunami and Internal Waves [in Russian], MGI, Sevastopol' (1976).
10. N. A. Zavol'skii, "Features of the propagation of linear internal waves in a continuously stratified liquid," Izv. Akad. Nauk SSSR, MZhG, No. 1 (1987).
11. R. G. Rehm and H. S. Radt, "Internal waves generated by a translation oscillating body," J. Fluid Mech., 68, No. 2 (1975).

PROPAGATION OF A BOUNDARY DISTURBANCE IN A STRATIFIED GAS FOR ARBITRARY KNUDSEN NUMBER

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Introduction. A systematic treatment of wave disturbances in rarefied gases should be based on the Boltzmann kinetic equation or its standard approximations [1, 2]. The purpose of the present paper is to use the kinetic equation to study the forced vibrations of a vertically stratified gas in a gravitational field for given types of excitation on the boundaries. Defining the Knudsen number Kn of the problem as the ratio of the mean free path of the gas molecules to the scale of the inhomogeneities due to the propagating waves, we find that Kn increases with height because of the height dependence of the mean free path in the stratified gas. Hence it is required to determine the motion of the gas for arbitrary Kn.

In many respects this problem is similar to the well-known problem of propagation of ultrasound in a uniform gas. Interest in the latter problem from the point of view of the kinetic theory of gases was stimulated by the work of Van Chan and Uhlenbeck [2]. Important results in this field have been obtained for the linearized Boltzman equation and for approximate kinetic equations using analytic continuation of dispersion relations [3], the Wiener–Hopf method [4], reduction to a Riemann–Hilbert problem [5, 6], and numerical integration along the characteristics [7]. These results suggest that the wave nature of disturbances persists in a gas with $\mathrm{Kn} \geqslant 1$. In this case the phase velocity and absorption coefficient of acoustic waves calculated with the help of the BGK kinetic equation are found to be in good agreement with experiment. The BGK equation can also be used to analyze the propagation of wave disturbances in a stratified gas. Physically, the stratification of the gas leads to internal waves, together with the usual acoustic waves. The dispersion relation for internal waves is quite different from the dispersion relation for acoustic waves and the study of kinetic effects on the propagation of internal waves is of interest in the physics of the upper atmosphere [8]. However, the presence of an external field and the stratification of the gas complicates the problem, since the result is an equation with variable coefficients. Hence the usual methods of finding the solution for sound in a uniform gas are no longer applicable, since they rely on separation of variables with the help of the Fourier transform. The method of integration along the characteristics has to be modified to take into account nonlinear characteristics.

To describe the propagation of boundary disturbances in a stratified gas for arbitrary Knudsen number we reduce the integrodifferential BGK equation to a closed system of integral equations for the first five moments of the distribution function. A general integral kinetic equation including the boundary conditions on the surface of a body in a flowing gas was obtained earlier in [9]. This equation was solved in [10] by transforming to a system of

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integral equations for the expansion coefficients of the distribution function in a series of generalized Hermite polynomials in velocity space. An important step in obtaining the results of the present paper is the use of an approximate form of the collision integral in order to close the system of moment equations for the gas in a gravitational field by integrating along parabolic characteristics.

Statement of the Problem. A monatomic gas is found in a gravitatinal field g and is stratified exponentially in z (along the vertical) in the equilibrium state. The gas is assumed to be bounded from below (at z = 0) and above (z = h) by planes whose motion generates the propagating disturbances. Then for the linear correction φ to the Maxwell–Boltzmann distribution

$$f_0 = n_0 (\sqrt{\pi}\, v_T)^{-3} \exp\left(-\frac{z}{H} - \frac{v^2}{v_T^2}\right) \tag{1}$$

we write down the linearized BGK equation

$$\frac{\partial\varphi}{\partial t} + \mathbf{v}\cdot\frac{\partial\varphi}{\partial\mathbf{r}} - g\frac{\partial\varphi}{\partial v_z} = \nu \exp\left(-\frac{z}{H}\right)\left\{\sum_{i=1}^{5} M_i(t, \mathbf{r})\, \chi_i(\mathbf{v}) - \varphi\right\} \tag{2}$$

with the boundary conditions

$$\varphi|_{z=0} = \varphi_0(t, x, y, \mathbf{v}), \quad \varphi|_{z=h} = \varphi_1(t, x, y, \mathbf{v}). \tag{3}$$

Here n_0 is the density of the unperturbed gas at z = 0, $v_T = (2k_BT_0/m)^{1/2}$ is the average thermal speed of the molecules, $H \equiv k_BT_0/mg$ is the scale height of the atmosphere, $v = (v_x, v_y, v_z)$ is the velocity of the gas molecules, r = (x, y, z) is the position vector, t is the time, $\nu = n_0k_BT_0/\eta$ is the collision frequency of the molecules (the parameter of the BGK model), and η is the shear viscosity of the gas. The functions χ_i on the right-hand side of (2) are the eigenfunctions of the linearized Boltzmann collision operator corresponding to zero eigenvalue:

$$\chi_1 = 1, \quad \chi_2 = \sqrt{2}\,\frac{v_x}{v_T}, \quad \chi_3 = \sqrt{2}\,\frac{v_y}{v_T}, \quad \chi_4 = \sqrt{2}\,\frac{v_z}{v_T}, \quad \chi_5 = \sqrt{\frac{2}{3}}\left(\frac{v^2}{v_T^2} - \frac{3}{2}\right). \tag{4}$$

The moments M_i are defined as the scalar products of χ_i with the weight f_0 given by (1):

$$M_i \equiv \langle\chi_i, \varphi\rangle = \frac{1}{\pi^{3/2} v_T^3}\int d\mathbf{v}\cdot\exp(-v^2/v_T^2)\, \chi_i\varphi \tag{5}$$

and are related to the hydrodynamic parameters of the gas (the density n, average velocity u, and temperature T):

$$M_1 = \frac{n - n_0}{n_0}, \quad M_2 = \sqrt{2}\,\frac{u_x}{v_T}, \quad M_3 = \sqrt{2}\,\frac{u_y}{v_T}, \quad M_4 = \sqrt{2}\,\frac{u_z}{v_T}, \quad M_5 = \sqrt{\frac{3}{2}}\,\frac{T - T_0}{T_0}. \tag{6}$$

Integral Form of the BGK Equation with the Boundary Conditions. In the spirit of the method of characteristics, we transform the integrodifferential equation (2) together with the boundary conditions (3) into an integral equation for the function φ. We introduce the new variables $\{\alpha'\} = \{t', \mathbf{r}', \mathbf{v}'\}$ and denote the old variables as $\{\alpha\} = \{t, \mathbf{r}, \mathbf{v}\}$. The direct and inverse transformations from $\{\alpha\}$ to $\{\alpha'\}$ are

$$\begin{array}{ll}
\{\alpha'(z)\}: & \{\alpha(z')\}: \\
z'(z) = z, & z(z') = z', \\
t'(z) = t + v_z/g, & t(z') = t' - \operatorname{sgn}(v_z')\sqrt{v_z'^2 - 2gz'}/g, \\
x'(z) = x + v_xv_z/g, & x(z') = x' - v_x'\operatorname{sgn}(v_z')\sqrt{v_z'^2 - 2gz'}/g, \\
y'(z) = y + v_yv_z/g, & y(z') = y' - v_y'\operatorname{sgn}(v_z')\sqrt{v_z'^2 - 2gz'}/g, \\
v_x'(z) = v_x, & v_x(z') = v_x', \\
v_y'(z) = v_y, & v_y(z') = v_y', \\
v_z'(z) = \operatorname{sgn}(v_z)\sqrt{v_z^2 + 2gz}, & v_z(z') = \operatorname{sgn}(v_z')\sqrt{v_z'^2 - 2gz'},
\end{array} \tag{7}$$

where we assume that $v_z'^2 - 2gz' \geqslant 0$. We note that the sgn function in (7) ensures that the transformation between $\{\alpha\}$ and $\{\alpha'\}$ is unique.

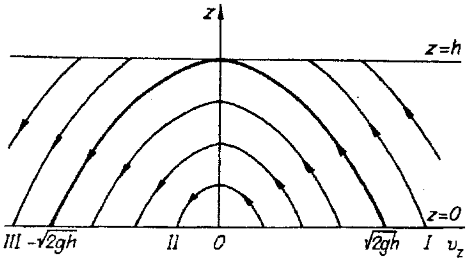

Fig. 1

In terms of the new variables $\{\alpha'\}$ the BGK equation (2) takes the form

$$v_z(z')\frac{\partial\varphi}{\partial z'} + \nu\exp\left(-\frac{z'}{H}\right)\varphi = \nu\exp\left(-\frac{z'}{H}\right)\sum_{i=1}^{5} M_i(t(z'), \mathbf{r}(z'))\,\chi_i(\mathbf{v}(z')). \tag{8}$$

Before formally integrating (8) with respect to z', we discuss the behavior of the characteristics of the left-hand side of (2), which are specified by $\alpha'(Z)$ from (7) at constant t', $\mathbf{r}'$, $\mathbf{v}'$. We construct these characteristics in the (z, v_z) plane.

We see from Fig. 1 that there are three types of characteristics for the problem of oscillations of a stratified gas between two planes. Characteristics of the first type begin from the plane z = 0 and end in the plane z = h. Characteristics of the second type begin at z = 0, change direction in the plane $v_z = 0$, and return to the plane z = 0. Characteristics of the third type begin at z = h and end at z = 0. The boundary separating the different types of characteristics (shown by the heavy line in Fig. 1) is given by the equation $v_z^2 = 2g(h - z)$.

In accordance with the form of the characteristics, the boundary functions φ_0 and φ_1 in velocity space are specified in terms of subregions in v_z: the function φ_0 (φ_1) is specified in three (two) subregions in v_z. These functions can be written in the form

$$\varphi|_{z=0} = \varphi_0 = \theta(v_z)\varphi_{0+} + \theta(-v_z)\theta(v_z + \sqrt{2gh})\varphi_{0-} + \theta(-v_z - \sqrt{2gh})\tilde{\varphi}_{0-}; \tag{9}$$

$$\varphi|_{z=h} = \varphi_1 = \theta(v_z)\varphi_{1+} + \theta(-v_z)\varphi_{1-}, \tag{10}$$

where $\theta(v_z)$ is the Heaviside unit step function.

The effect of the boundary functions on the right-hand sides of (9) and (10) propagate along the characteristics up to the opposite boundary. Therefore, all of these functions are related to one another. Of the five boundary functions in (9) and (10) really only two (φ_{0+} and φ_{1-}), corresponding to particles leaving the boundaries, can be specified as independent functions. These functions play an active role in the boundary conditions and are completely determined by the nature of the motion of the boundaries and the interaction of the boundaries with the gas. The three remaining "passive" functions (φ_{0-}, $\tilde{\varphi}_{0-}$, φ_{1+}), which take into account the contribution of particles incident on the boundaries, can be expressed in terms of φ_{0+} and φ_{1-}. It will be shown below that in the collisionless case the passive functions can be expressed in terms of the "active" functions by explicit linear relations. However the presence of collisions leads to integral terms in these relations which involve the total distribution function φ.

Equation (8) together with the boundary conditions (9) and (10) can be integrated with respect to z' along the characteristics, assuming that the boundary functions in (9) and (10) are formally independent. The region of velocity space in which φ is defined is broken up into three subregions (see Fig. 1). The direction of the integration along the characteristics in these subregions is chosen as follows. In regions I and II the integration goes from the boundary z' = 0 and in region III from z' = h. Combining the relations obtained in the three subregions of velocity space into a single relation and transforming back to the old variables $\{\alpha\}$, we have

$$\begin{aligned}\varphi(t, \mathbf{r}, \mathbf{v}) = \varphi_s(t, \mathbf{r}, \mathbf{v}) + \theta(v_z + \sqrt{2g(h-z)})\,\nu\int_0^z dz'\Phi(z, z', v_z)\,\times \\ \times\sum_{i=1}^{5} M_i(\{\alpha_1(z-z')\}, z')\,\chi_i(\mathbf{v}'(z-z')) - \theta(-v_z - \sqrt{2g(h-z)})\,\nu\,\times \\ \times\int_z^h dz'\,\Phi(z, z', v_z)\sum_{i=1}^{5} M_i(\{\alpha_1(z-z')\}; z')\,\chi_i(\mathbf{v}'(z-z')),\end{aligned} \tag{11}$$

where

$$\varphi_s(t, \mathbf{r}, \mathbf{v}) = \Psi(z, 0, v_z)\,[\theta(v_z)\,\varphi_{0+}(\{\alpha_1(z)\}, \mathbf{v}'(z)) + \theta(-v_z)\,\theta(v_z + \sqrt{2g(h-z)})\,\varphi_{0-}(\{\alpha_1(z)\}, \mathbf{v}'(z))] + \\ + \Psi(z, h, v_z)\,\theta(-v_z - \sqrt{2g(h-z)})\,\varphi_{1-}(\{\alpha_1(z-h)\}, \mathbf{v}'(z-h)) \tag{12}$$

and where we have introduced the functions

$$\Psi(z, \tilde{z}, v_z) = \exp\left\{-\nu \int_{\tilde{z}}^{z} dz'' \frac{\exp(-z''/H)}{v_z'(z-z'')}\right\}; \tag{13}$$

$$\Phi(z, \tilde{z}, v_z) = \Psi(z, \tilde{z}, v_z)\exp(-\tilde{z}/H)/v_z'(z-\tilde{z}). \tag{14}$$

In (13) and (14) the functions $v_z'(z - z'')$ and $v_z'(z - \tilde{z})$ are defined as before by (7) and the set of independent variables $\{\alpha_1(z)\} = \{t_1(z), x_1(z), y_1(z)\}$ in (12) is given by

$$t_1(z) = t'(z) - \frac{v_z'(z)}{g}, \quad x_1(z) = x'(z) - \frac{v_x v_z'(z)}{g}, \quad y_1(z) = y'(z) - \frac{v_y v_z'(z)}{g}. \tag{15}$$

The functions $\tilde{\varphi}_{0-}$ and φ_{1+} do not appear explicitly in (12), since the integration is carried out from the boundary z = 0 along the characteristics of the first and second types and from z = h along characteristics of the third type. In transforming the BGK equation to integral form a different choice of the direction of integration along the characteristics in regions I and III is possible. In this case the functions φ_s would contain different boundary functions from (9) and (10), but the number of boundary functions would still be equal to three. When the dependence of the passive functions on the active functions is taken into account, all possible integral forms of the BGK equation are equivalent.

Since the moments M_i are related to φ by (5), Eq. (11) is an integral equation for the distribution function φ. It is equivalent to (2) with the boundary conditions (9) and (10), which can be verified by direct differentiation with respect to t, **r**, and v_z.

We show that (11) satisfies the boundary conditions (9) and (10). We consider (11) when $v_z > -(2g(h - z))^{1/2}$ and $z \to 0$. With the help of (12) we have

$$\varphi = \theta(v_z)\,\varphi_{0+}(t, x, y, \mathbf{v}) + \theta(-v_z)\,\varphi_{0-}(t, x, y, \mathbf{v}).$$

Hence in this case (11) satisfies the lower boundary condition. When $v_z < -(2g(h - z))^{1/2}$ and $z \to 0$ we have $\varphi = \tilde{\varphi}_{0-}$ from (9). But comparing (9) and the relation obtained from (11) and (12) for these conditions, we obtain the following relation between $\tilde{\varphi}_{0-}$ and φ_{1-}

$$\tilde{\varphi}_{0-} = \varphi_{1-}(\{\alpha_1(-h)\}, \mathbf{v}_1(-h))\,\Psi(0, h, v_z) - \nu \int_0^h dz''\Phi(0, z'', v_z) \sum_{i=1}^{5} M_i(\{\alpha_1(-z'')\}, z'')\,\chi_i(\mathbf{v}'(-z'')). \tag{16}$$

Similarly when $v_z > 0$ and $z \to h$ we obtain a relation between φ_{1+} and φ_{0+}

$$\varphi_{1+} = \Psi(h, 0, v_z)\,\varphi_{0+}(\{\alpha_1(h)\}, \mathbf{v}'(h)) + \nu \int_0^h dz''\Phi(h, z'', v_z) \sum_{i=1}^{5} M_i(\{\alpha_1(h-z'')\}, z'')\,\chi_i(\mathbf{v}'(h-z'')). \tag{17}$$

Finally when $v_z < 0$ and $z \to h$ we obtain from (11) and (12) that $\varphi = \varphi_{1-}(t, x, y, \mathbf{v})$, which is the upper boundary condition (10).

Motion of the Gas in Limiting Cases. We consider the motion of the gas in different limiting cases of (11): free-molecular flow, the limit of no stratification, and propagation of disturbances in a stratified atmosphere bounded only from below at z = 0.

We consider first free-molecular flow of a gas. Setting the collision frequency $\nu = 0$ in (11), we obtain

$$\varphi = \theta(v_z)\,\varphi_{0+}(\{\alpha_1(z)\}, \mathbf{v}'(z)) + \theta(-v_z)\,\theta(v_z + \sqrt{2g(h-z)}) \times \\ \times \varphi_{0-}(\{\alpha_1(z)\}, \mathbf{v}'(z)) + \theta(-v_z - \sqrt{2g(h-z)})\,\varphi_{1-}(\{\alpha_1(z-h)\}, \mathbf{v}'(z-h)). \tag{18}$$

We see from (18) that for free-molecular flow the state of the gas at the point t, **r**, **v** is determined for $v_z > 0$ by particles leaving the surface z = 0 at time $t + \frac{1}{g}[v_z - \sqrt{v_z^2 + 2gz}]$ with velocities $v_x, v_y, \sqrt{v_z^2 + 2gz}$. When $-(2g(h - z))^{1/2} < v_z < 0$ the state of the gas is determined by particles arriving at the surface z = 0 at time $t - \frac{1}{g}[|v_z| - \sqrt{v_z^2 + 2gz}]$ at the point whose coordinates are $x - v_x\,[|v_z| - \sqrt{v_z^2 + 2gz}\,]/g$, $y - v_y\,[|v_z| - \sqrt{v_z^2 + 2gz}\,]/g$ and with the velocities v_x, v_y, $-\sqrt{v_z^2 + 2gz}$. These particles left the surface z = 0, reversed direction in the gravitational field, and returned to z = 0 without reaching z = h. When $v_z < -(2g(h - z))^{1/2}$ the state of gas is determined by particles leaving the boundary z = h at time $t - \frac{1}{g}\,[|v_z| - \sqrt{v_z^2 + 2g\,(z - h)}\,]$ from the point whose coordinates are $x - v_x\,[|v_z| - \sqrt{v_z^2 + 2g\,(z - h)}\,]/g$, $y - v_y\,[|v_z| - \sqrt{v_z^2 + 2g\,(z - h)}\,]/g$ with velocity v_x, v_y, $-\sqrt{v_z^2 + 2g\,(z - h)}$. As expected, for a particle leaving the boundary z = h its velocity is always negative at any point z < h.

We next consider a gas without stratification. Taking the limits $H \to \infty$ and $g \to 0$ in (11)-(15), we find

$$\varphi = \theta\,(v_z)\exp\left(-\nu\frac{z}{v_z}\right)\varphi_{0+}\,(\{\tilde\alpha\,(z)\},\,\mathbf{v}) + \theta\,(-v_z)\exp\left(\nu\frac{h-z}{v_z}\right)\times$$
$$\times\,\varphi_{1-}\,(\{\tilde\alpha\,(h-z)\},\,\mathbf{v}) + \theta\,(v_z)\,\nu\int_0^z dz'\,\frac{1}{v_z}\exp\left(-\nu\frac{z-z'}{v_z}\right)\sum_{i=1}^{5} M_i\,(\{\tilde\alpha\,(z-z')\},\,z')\,\times$$
$$\times\,\chi_i\,(\mathbf{v}) - \theta\,(-v_z)\,\nu\int_z^h dz'\,\frac{1}{v_z}\exp\left(-\nu\frac{z-z'}{v_z}\right)\sum_{i=1}^{5} M_i\,(\{\tilde\alpha\,(z-z')\},\,z')\,\chi_i\,(\mathbf{v}), \tag{19}$$

where $\{\tilde\alpha\,(z)\} = \{t - z/v_z,\ x - v_x z/v_z,\ y - v_y z/v_z\}$.

If in (19) the distance h between the boundaries goes to infinity then

$$\varphi = \theta\,(v_z)\exp\left(-\nu\frac{z}{v_z}\right)\varphi_{0+}\,(\{\tilde\alpha\,(z)\},\,\mathbf{v}) + \theta\,(v_z)\,\nu\int_0^z dz'\,\frac{1}{v_z}\exp\left(-\nu\frac{z-z'}{v_z}\right)\sum_{i=1}^{5} M_i\,(\{\tilde\alpha\,(z-z')\},\,z')\,\chi_i\,(\mathbf{v}) -$$
$$-\,\theta\,(-v_z)\,\nu\int_z^\infty dz'\,\frac{1}{v_z}\exp\left(-\nu\frac{z-z'}{v_z}\right)\sum_{i=1}^{5} M_i\,(\{\tilde\alpha\,(z-z')\},\,z')\,\chi_i\,(\mathbf{v}). \tag{20}$$

In the one-dimensional case (20) reduces exactly to the equation obtained in [4, 7] for propagation of one-dimensional sound in a uniform unbounded atmosphere. The effect of the upper boundary vanishes when $h \to \infty$, since the exponential in (19) in front of φ_{1-} vanishes at large h and $v_z < 0$. However from (16) it follows that in this case $\tilde\varphi_{0-} \neq 0$.

We consider the final limiting case when the upper boundary goes to infinity for a stratified gas. Putting H = const and taking the limit $h \to \infty$, we find from (11)

$$\varphi = \Psi\,(z,\,0,\,v_z)\,[\theta\,(v_z)\,\varphi_{0+}\,(\{\alpha_1\,(z)\},\,\mathbf{v}'\,(z)) + \theta\,(-v_z)\,\varphi_{0-}\,(\{\alpha_1\,(z)\},\,\mathbf{v}'\,(z))\,] +$$
$$+\,\nu\int_0^z dz'\,\Phi\,(z,\,z',\,v_z)\sum_{i=1}^{5} M_i\,(\{\alpha_1\,(z-z')\},\,z')\,\chi_i\,(\mathbf{v}'\,(z-z')). \tag{21}$$

This equation will be used as the basis for further study.

Relation between the Boundary Functions for Incident and Departing Particles. If we attempt to obtain the limit of a uniform gas by putting $H \to \infty$ in (21), we do not obtain (20) Since the relation between the passive boundary function φ_{0-} and the active function φ_{0+} must be taken into account in (21) before taking the limit $H \to \infty$.

We consider the relations between the boundary functions φ_{0-} and φ_{0+} in more detail. From Fig. 1 the function φ_{0-} (the velocity region $-\sqrt{2gh} < v_z < 0$) occurs on characteristics beginning and ending on the boundary z = 0 and not reaching z = h. Hence the condition for φ_{0-} should be the continuity condition of the distribution function on the characteristics at $v_z = 0$:

$$\varphi\,(t,\,\mathbf{r},\,\mathbf{v})\big|_{v_z=+0} = \varphi\,(t,\,\mathbf{r},\,\mathbf{v})\big|_{v_z=-0}. \tag{22}$$

Putting the right-hand side of (21) with $v_z = +0$ and $v_z = -0$ into (22), we obtain φ_{0-} in terms of φ_{0+}:

$$\varphi_{0-}\,(t,\,x,\,y,\,\mathbf{v})\big|_{v_z<0} = \Psi_-^2\,(0,\,v_z^2/2g,\,v_z)\,\varphi_{0+}\,(\{\alpha_1^+\,(0)\},\,\mathbf{v}'\,(0)) +$$
$$+\,\Psi_-\,(0,\,v_z^2/2g,\,v_z)\,\nu\sum_{i=1}^{5}\int_0^{v_z^2/2g} dz''\,\Delta^\pm\,\{\Phi\,(0,\,z'',\,v_z)\,\times$$
$$\times\,\Psi^{-1}\,(0,\,v_z^2/2g,\,v_z)\,M_i\,(\{\alpha_1\,(-z'')\},\,z'')\,\chi_i\,(\mathbf{v}'\,(z''))\}. \tag{23}$$

Here $\Delta^{\pm}(Q) \equiv Q_+ - Q_-$, the indices + and − determine the sign of the function $\mathrm{sgn}(v_z)$ appearing in (7), (13)-(15) for $\mathbf{v}'(z)$, Ψ, Φ, $\{\alpha_1\}$, respectively. Using (23), we can write (21) in the form

$$\varphi(\mathbf{r}, \mathbf{v}, t) = [\theta(v_z) + \theta(-v_z)\Psi_-^2(z, z + v_z^2/2g, v_z)]\Big[\Psi_+(z, 0, v_z) \times$$
$$\times \varphi_{0+}(\{\alpha_1^+(z)\}, \mathbf{v}'^+(z)) + \nu \int_0^z dz''\Phi_+(z, z'', v_z) \sum_{i=1}^5 M_i(\{\alpha_1^+(z - z'')\}, z'') \times$$
$$\times \chi_i(\mathbf{v}'^+(z - z''))\Big] + \theta(-v_z)\Psi_-(z, z + v_z^2/2g, v_z)\, \nu \sum_{i=1}^5 \int_z^{z+v_z^2/2g} dz''\Delta^{\pm} \times$$
$$\times \{\Phi(z, z'', v_z)\Psi^{-1}(z, z + v_z^2/2g, v_z) M_i(\{\alpha_1(z - z'')\}, z'')\chi_i(\mathbf{v}'(z - z''))\}. \qquad (24)$$

Note that if we take the limit of a uniform gas in (24) by putting $H \to \infty$, $g \to 0$, we obtain (20).

It is not difficult to extend these results to the case of an upper boundary at $z = h$. In the presence of the upper boundary the continuity condition (22) is imposed on the distribution function only along the characteristics of the second type (see Fig. 1) in the region $0 \leqslant z \leqslant h$.

The continuity condition (22), and hence (23), are general relations holding for any form of the function φ_{0+}. For further study we consider diffuse reflection from a plate oscillating about $z = 0$ with the vertical (along the z axis) velocity $u_0(t, x, y)$:

$$f(t, \mathbf{r}, \mathbf{v})|_{v_z>0, z=0} = n_s(\sqrt{\pi}\, v_T)^{-3} \exp\left\{-\frac{(\mathbf{v} - \mathbf{u})^2}{v_T^2}\right\}. \qquad (25)$$

In this problem the density n_s corresponds to the steady-state motion of the gas. According to the usual procedure [4-7], n_s is determined by requiring that the average vertical velocity at $z = 0$ be equal to $u_0(t, x, y)$. Linearizing (25), we find

$$\varphi_{0+}(t, x, y, \mathbf{v}) = \varphi|_{v_z>0, z=0} = 2\frac{u_0}{v_T^2} v_z + \frac{n_s - n_0}{n_0}. \qquad (26)$$

Substituting (26) into (5) for $M_n|_{z=0}$, we obtain for $(n_s - n_0)/n_0 \equiv M_{1s}$

$$M_{1s} = \frac{\sqrt{\pi}\, u_0}{v_T} - \frac{2}{\pi v_T^4} \int_{v_z<0} d\mathbf{v} \exp(-v^2/v_T^2)\, v_z \varphi|_{z=0, v_z<0}. \qquad (27)$$

Therefore, (26) and (27) relate the distribution functions belonging to the different half spaces in v_z. The function $\varphi_{0-} = \varphi|_{z=0, v_z<0}$ in (27) should be determined using (23).

System of Integral Equations for the Moments of the Distribution Function. The characteristic feature of the BGK equation is that the distribution function can be found in terms of its first five moments. Indeed, if the moments $M_1, \ldots, M_5$ are known we can substitute them in (24) and integrate with respect to z, thereby obtaining the function φ. The contribution of the higher moments to the distribution function is effectively taken into account by the kinetic boundary condition on the right-hand side of (24) and by the fact that in performing the integration with respect to z the quantities M_i ($i = 1, \ldots, 5$) must be known in all space and for all previous times.

We put $\mathbf{u}_0 = (0, 0, \tilde{u}_0 \exp(i\omega t - ik_x x - ik_y y))$ and look for the solution to (24) in the form

$$\varphi(t, \mathbf{r}, \mathbf{v}) = \tilde{\varphi}(z, \mathbf{v}) \exp(i\omega t - ik_x x - ik_y y). \qquad (28)$$

We then have

$$\varphi_{0+}(t, x, y, \mathbf{v}) = \tilde{\varphi}_{0+}(\mathbf{v}) \exp(i\omega t - ik_x x - ik_y y); \qquad (29)$$
$$M_n(t, \mathbf{r}) = \tilde{M}_n \exp(i\omega t - ik_x x - ik_y y), \qquad (30)$$

and the quantities $\tilde{M}_n$ and $\tilde{\varphi}$ are related as before by (5).

We next obtain from (24) a system of integral equations for the moments $\tilde{M}_n$. We take the scalar product of (24) with the eigenfunctions χ_n of (4), corresponding to these moments. It will be convenient to transform to dimensionless variables

$$\mathbf{c} \equiv \mathbf{v}/v_T, \quad \zeta \equiv z/H. \qquad (31)$$

Using the identity

$$\int\limits_{c_z<0} dc \int\limits_{\zeta}^{\zeta+c_z^2/2g} d\zeta' \ldots = \int\limits_{\zeta}^{\infty} d\zeta' \int\limits_{c_z<\sqrt{\zeta'-\zeta}} dc \ldots,$$

we obtain from (24) and (28)-(30) a system of inhomogeneous Fredholm integral equations of the second kind for the $\tilde{M}_n$:

$$\tilde{M}_n(\zeta) = P_n(\zeta) + \tilde{M}_{1s}R_n(\zeta) + \sum_{m=1}^{5}\int_0^{\zeta} d\zeta' K_{nm}^{(1)}(\zeta, \zeta')\tilde{M}_m(\zeta') + \sum_{m=1}^{5}\int_{\zeta}^{\infty} d\zeta' K_{nm}^{(2)}(\zeta, \zeta')\tilde{M}_m(\zeta'). \tag{32}$$

The functions $P_n(\zeta)$, $R_n(\zeta)$, $K_{nm}^{(1)}(\zeta, \zeta')$, $K_{nm}^{(2)}(\zeta, \zeta')$ are conveniently expressed in terms of the two auxiliary functions

$$I_{n,m}(\zeta, \zeta') = \sqrt{\pi}\int_0^{\infty} dc \exp\{-c^2 + g^+(\zeta, \zeta', c)\} c^n (c^2 + \zeta')^{m/2}; \tag{33}$$

$$J_{n,m}(\zeta, \zeta') = \sqrt{\pi}\int_0^{\infty} dc \exp\{-c^2 + g^-(\zeta, \zeta', c) - 2g^-(\zeta, -c^2, c)\} c^n (c^2 + \zeta')^{m/2}. \tag{34}$$

Here

$$g^{\pm}(\zeta, \zeta', c) = -\int_0^{\zeta'} d\zeta_1 \, [\delta_v(\zeta)\exp(\zeta') + i\delta_\omega \mp \delta^2 c](c^2 - \zeta_1)^{-1/2};$$

$$\delta_v(\zeta) = \frac{vH}{v_T}\exp(-\zeta); \quad \delta_\omega = \frac{\omega H}{v_T}; \quad \delta^2 = \delta_x^2 + \delta_y^2 = H^2(k_x^2 + k_y^2).$$

In the one-dimensional case ($\delta^2 = 0$) we have $g^+ = g^-$. The functions $I_{n,m}$ and $J_{n,m}$ obey the recursion relations

$$\begin{gathered} I_{n,m+2}(\zeta, \zeta') = I_{n+2,m}(\zeta, \zeta') + \zeta' I_{n,m}(\zeta'), \\ I_{n,m+4}(\zeta, \zeta') = I_{n+4,m}(\zeta, \zeta') + 2\zeta' I_{n+2,m}(\zeta') + \zeta'^2 I_{n,m}(\zeta, \zeta'), \\ I_{n,m+2p}(\zeta, \zeta') = \sum_{k=0}^{p} C_p^k I_{n+2(p-k),m}\zeta'^k \end{gathered} \tag{35}$$

and the following relation for the derivative with respect to the argument ζ':

$$\frac{\partial}{\partial\zeta'} I_{n,m}(\zeta, \zeta') = -[\delta_v(\zeta)\exp(\zeta') + i\delta_\omega] I_{n,m-1}(\zeta, \zeta') + \frac{m}{2} I_{n,m-2}(\zeta, \zeta') + \delta^2 I_{n+1,m-1}(\zeta, \zeta'). \tag{36}$$

The relations (35) and (36) have the same form for $J_{n,m}$. A relation for the derivative of $I_{n,m}$ with respect to ζ can also be obtained, but it is not written out here because of its complexity.

We also note that in the limit of a uniform gas $g \to 0$, $H \to \infty$, the function $J_{n,m}(\zeta, \zeta') \to 0$ and $I_{n,m}(\zeta, \zeta')$ reduces to the Abramowitz function [11] well known in the kinetic theory of uniform gases.

In the one-dimensional case ($\delta^2 = 0$), the quantities in (32) can be rewritten as (n, m = 1, 2, 3)

$$P_1(\zeta) = 2\frac{u_0}{v_T}[I_{0,1} + J_{0,1}](\zeta, \zeta), \quad P_2(\zeta) = 2\frac{u_0}{v_T}\sqrt{2}\,[I_{1,1} - J_{1,1}](\zeta, \zeta),$$

$$P_3(\zeta) = 2\frac{u_0}{v_T}\sqrt{\frac{2}{3}}\,[I_{2,1} - \frac{1}{2}I_{0,1} + J_{2,1} - \frac{1}{2}J_{0,1}](\zeta, \zeta),$$

$$R_1(\zeta) = [I_{0,0} + J_{0,0}](\zeta, \zeta), \quad R_2(\zeta) = \sqrt{2}\,[I_{1,0} - J_{1,0}](\zeta, \zeta),$$

$$R_3(\zeta) = \sqrt{\frac{2}{3}}\,[I_{2,0} - \frac{1}{2}I_{0,0} + J_{2,0} - \frac{1}{2}J_{0,0}](\zeta, \zeta), \quad K_{1,1}^{(1)}(\zeta, \zeta') = \delta_v(\zeta')\,[I_{0,-1} + J_{0,-1}](\zeta, \zeta - \zeta'),$$

$$K_{1,2}^{(1)}(\zeta, \zeta') = \delta_v(\zeta')\sqrt{2}\,[I_{0,0} + J_{0,0}](\zeta, \zeta - \zeta'),$$

$$K_{1,3}^{(1)}(\zeta, \zeta') = \delta_v(\zeta')\sqrt{\frac{2}{3}}\,[I_{0,1} - \frac{1}{2}I_{0,-1} + J_{0,1} - \frac{1}{2}J_{0,-1}](\zeta, \zeta - \zeta'),$$

$$K_{2,1}^{(1)}(\zeta, \zeta') = \delta_v(\zeta')\sqrt{2}\,[I_{1,-1} - J_{1,-1}](\zeta, \zeta - \zeta'),$$

$$K_{2,2}^{(1)}(\zeta, \zeta') = \delta_v(\zeta')\,2\,[I_{1,0} - J_{1,0}](\zeta, \zeta - \zeta'),$$

$$K_{2,3}^{(1)}(\zeta, \zeta') = \delta_v(\zeta')\frac{2}{\sqrt{3}}\,[I_{1,1} - \frac{1}{2}I_{1,-1} - J_{1,1} - \frac{1}{2}J_{1,-1}](\zeta, \zeta - \zeta'),$$

$$K_{3,1}^{(1)}(\zeta, \zeta') = \delta_v(\zeta')\sqrt{\frac{2}{3}}\,[I_{2,-1} - \frac{1}{2}I_{0,-1} + J_{2,-1} - \frac{1}{2}J_{0,-1}](\zeta, \zeta - \zeta'),$$

$$K_{3,2}^{(1)}(\zeta, \zeta') = \delta_v(\zeta') \frac{2}{\sqrt{3}} [I_{2,0} - \frac{1}{2} I_{0,0} + J_{2,0} - \frac{1}{2} J_{0,0}] (\zeta, \zeta - \zeta'),$$

$$K_{3,3}^{(1)}(\zeta, \zeta') = \delta_v(\zeta') \frac{2}{3} [\frac{5}{4} I_{0,-1} - \frac{1}{2} I_{0,1} - \frac{1}{2} I_{2,-1} + I_{2,1} + \frac{5}{4} J_{0,-1} - \frac{1}{2} J_{0,1} - \frac{1}{2} J_{2,-1} + J_{2,1}] (\zeta, \zeta - \zeta').$$

To shorten the notation, in these expressions the arguments common to all of the functions $I_{n,m}$ and $J_{n,m}$ have been written in parentheses to the right of the square brackets. Expressions for $K_{n,m}^{(2)}(\zeta, \zeta')$ are easily obtained from the relations for $K_{n,m}^{(1)}(\zeta, \zeta')$ with the help of the substitutions

$$I_{n,m}(\zeta, \zeta - \zeta') \to (-1)^{n+m} I_{m+1,n-1}(\zeta', \zeta' - \zeta), \quad J_{n,m}(\zeta, \zeta - \zeta') \to J_{m+1,n-1}(\zeta', \zeta' - \zeta).$$

The above relations form a complete set of expressions for P_n, R_n, $K_{n,m}^{(1)}$, $K_{n,m}^{(2)}$ in terms of $I_{n,m}$ and $J_{n,m}$.

The system of equations (32) also involves $\tilde{M}_{1s}$, which is a functional of the distribution function for $c_z < 0$ and $\zeta = 0$. According to [7], this difficulty can be overcome by assuming a solution to (32) of the form

$$\tilde{M}_n(\zeta) = a_n(\zeta) + \tilde{M}_{1s} b_n(\zeta). \tag{37}$$

Substituting (37) into (32), we obtain the equations

$$a_n(\zeta) = P_n(\zeta) + \sum_{m=1}^{5} \int_0^{\zeta} d\zeta' K_{nm}^{(1)}(\zeta, \zeta') a_m(\zeta') + \sum_{m=1}^{5} \int_{\zeta}^{\infty} d\zeta' K_{nm}^{(2)}(\zeta, \zeta') a_m(\zeta'); \tag{38}$$

$$b_n(\zeta) = R_n(\zeta) + \sum_{m=1}^{5} \int_0^{\zeta} d\zeta' K_{nm}^{(1)}(\zeta, \zeta') b_m(\zeta') + \sum_{m=1}^{5} \int_{\zeta}^{\infty} d\zeta' K_{nm}^{(2)}(\zeta, \zeta') b_m(\zeta'), \tag{39}$$

which do not involve $\tilde{M}_{1s}$.

Solving (38) and (39) for the functions a_n and b_n, and thereby finding $\tilde{\varphi}$ for $\zeta = 0$ and $c_z < 0$ from (23), we substitute the result into (27) for $\tilde{M}_{1s}$. Using (37), we obtain a linear algebraic equation for $\tilde{M}$ whose solution is

$$\tilde{M} = [\frac{2\tilde{u}_0}{v_T} S_1 - T\{a\}] [S_0 + T\{b\}]^{-1}, \tag{40}$$

where

$$S_p = [I_{1,p} - J_{1,p}](0, 0); \tag{41}$$

$$T\{a\} = \sum_{m=1}^{3} \int_0^{\infty} d\zeta' K_{2,m}^{(2)}(0, \zeta') a_m(\zeta'). \tag{42}$$

The system of equations (33)-(42) represents an algorithm for solving (32) for the moments of the distribution function and therefore the integral equation (24) for the distribution function φ itself in the case of wave disturbances generated by an oscillating plane in a monatomic gas stratified in a gravitational field. A crucial step in this algorithm is the calculation of the functions $I_{n,m}(\zeta, \zeta')$ and $J_{n,m}(\zeta, \zeta')$, which are analogous to the Ambramowitz functions in the approach of [7] to the propagation of a boundary disturbance in a uniform gas. In the first step of the procedure, (38) and (39) are solved, taking into account (33)-(36). The solutions are substituted into (40) and $\tilde{M}_{1s}$ is determined with the help of (41) and (42). The moments $\tilde{M}_n$ are then found using (37). Finally, they can be substituted into (24) for φ.

LITERATURE CITED

1. C. Cercignani, Theory and Application of the Boltzmann Equation, Elsevier, North-Holland–New York (1976).
2. G. E. Uhlenbeck and G. W. Ford, Lectures in Statistical Mechanics, American Mathematical Society, Providence, RI (1963).
3. L. Sirovich and J. K. Thurber, "Plane-wave propagation in kinetic theory," J. Math. Phys., 8, No. 4 (1967).
4. T. G. Richardson and L. Sirovich, "The sound-wave boundary-value problem in kinetic theory," J. Math. Phys., 12, No. 8 (1971).

5. J. P. Thomas, Jr. and C. E. Siewert, "Sound-wave propagation in a rarefied gas," Trans. Theory Stat. Phys., 8, 219 (1979).
6. K. Aoki and C. Cercignani, "A technique for time-dependent boundary-value problems. 2. Application to sound propagation," ZAMP, 35, No. 3 (1984).
7. S. K. Loyalka and T. C. Cheng, "Sound-wave propagation in a rarefied gas," Phys. Fluids, 22, No. 5 (1979).
8. A. K. Shchekin, S. B. Leble, and D. A. Vereshchagin, Introduction to the Physical Kinetics of Rarefied Gases [in Russian], Kaliningrad State University (1990).
9. S. V. Vallander, "New kinetic equations in the theory of monatomic gases," Dokl. Akad. Nauk SSSR, 131, No. 1 (1960); "Problems and equations for aerodynamics of rarefied gases," in: Aerodynamics of Rarefied Gases [in Russian], No. 1, Leningrad State University (1963).
10. R. G. Barantsev, "Method of integral moment kinetic equations," Dokl. Akad. Nauk SSSR, 151, No. 5 (1963).
11. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, New York (1964).